

An improved effective field theory formulation of spin-1 Ising systems with arbitrary coordination number z

Ümit Akıncı¹

Department of Physics, Dokuz Eylül University, TR-35160 Izmir, Turkey

1 Abstract

An improved unified formulation based on the effective field theory is introduced for a spin-1 Ising model with nearest neighbor interactions with arbitrary coordination number z . Present formulation is capable of calculating all the multi-spin correlations systematically in a representative manner, as well as its single site counterparts in the system and gives much better results for critical temperature, in comparison with the other works in the literature. The formulation can be easily used to various kinds of spin-1 Ising models, as long as the system contains only the nearest neighbor interactions as spin-spin interactions. **Keywords: spin-1 Ising model ; Effective field theory ; Correlation functions; Bond diluted Ising system; Random field distributions, Crystal field dilution**

2 Introduction

Spin-1 Blume-Capel (BC) model [1,2] is one of the most extensively studied models in statistical mechanics. Beside the various real magnetic systems, the model or some extensions can describe many physical systems such as ternary alloys and multicomponent fluids such as ${}^3\text{He} - {}^4\text{He}$ mixtures [3]. Although the BC model is a very simple model, it exhibits many multicritical phenomena such as first (second) order order-disorder phase transitions, tricritical points. On the other hand by inclusion of random distributions of crystal field and/or external longitudinal magnetic field or bond dilution of the BC model, makes the model more valuable. For example, BC model with random crystal field can describe ${}^3\text{He} - {}^4\text{He}$ mixtures in a random media i.e. aerogel [4]. Taking into account these randomness effects will change phase diagrams of the model drastically, then model would exhibit richer multicritical phenomena.

The spin-1 BC model was studied in different lattices by a variety of methods such as two spin cluster approximation [5], Bethe lattice approximation [6], series expansion method [7, 8], cluster variation method [9, 10], Monte Carlo simulations [11–14], renormalization group [15–17] and effective field theory (EFT) [18–20]. In order to simplify the application and improve the results of EFT, Kaneyoshi introduced differential operator technique [21–25] and Du introduced the expanded Bethe-Peierls approximation (BPA) [26–28].

On the other hand there exist several works dealing with the BC model with random crystal field and bond dilution. Among these works, [29–33] examined the random crystal field system with EFT. Some other applied techniques about this system are, mean field theory [34–40], cluster variation method [41], Bethe lattice approximation [42], finite cluster approximation [43, 44], Monte Carlo simulation [45], pair approximation [46] and renormalization group method [47]. Also the bond dilution problem in (transverse field) BC model was investigated with similar methods such as EFT [48–60], finite cluster approximation [61–65], Monte Carlo simulations [66], series expansion [67], two spin cluster approximation [68–70] and renormalization group method [71].

As seen in this very short literature about the spin-1 BC model and some of its extensions, the EFT method for solutions of this model has an important place in the literature. Thus, the aim of this work is to introduce an improved general effective field formulation for the spin-1 BC model and its variants, which is capable of calculating all multi site correlations of the system, in contrast to the other EFT formalisms in the related literature.

A typical EFT method or finite cluster approximation start by choosing a finite representative cluster and using exact spin identity (e.g. given in [72]) which gives the thermal average of magnetization. In the process of evaluating this identity in the cluster, multi site spin correlations

¹umit.akinci@deu.edu.tr

appear. Almost all of the works in the literature treat these multi site spin correlations by using a decoupling approximation (DA) which completely neglects them. Recently we have treated these multi site spin correlations in the spin-1 BC model [73–75] for the specific lattices and showed that taking into account these correlations considerably improves the results. Beside the improvement of the results, calculation of these correlations allows us to obtain some thermodynamic quantities more accurately (e.g internal energy, specific heat).

The aim of this work is to obtain a general EFT formulation via treating the multi site spin correlations which is valid for spin-1 BC model and its variants for any arbitrary coordination number, as long as the system contains only nearest neighbor interactions as spin-spin interactions. For this purpose, the paper is organized as follows: In Sec. 3 we present this general formulation. As an application of our formulation we give the results of the pure, bond diluted and crystal field diluted spin-1 BC model on three dimensional lattices in Sec. 4, and finally Sec. 5 contains our conclusions.

3 Model and Formulation

The Hamiltonian of our system is given by

$$\mathcal{H} = - \sum_{\langle i,j \rangle} J_{ij} s_i s_j - \sum_i D_i s_i^2 - \sum_i H_i s_i \quad (1)$$

where s_i is the z component of the spin and it takes the values $s_i = 0, \pm 1$ for the spin-1 system, $J_{ij} = J_{ji} > 0$ is the ferromagnetic exchange interaction between spins i and j , D_i is the crystal field, H_i is the external longitudinal magnetic field at a lattice site i . The first summation in Eq. (1) is over the nearest-neighbor pairs of spins and the other summations are over the all lattice sites. J_{ij} , D_i and H_i may be given with certain distributions or they can have the same values for all pairs/sites, i.e $J_{ij} = J, D_i = D, H_i = H$.

We consider a lattice which has N identical spins arranged. We define a cluster on the lattice which consists of a central spin labeled s_0 , and z perimeter spins being the nearest neighbors of the central spin. The nearest-neighbor spins are in an effective field produced by the outer spins, which can be determined by the condition that the thermal average of the central spin is equal to that of its nearest-neighbor spins.

We start with spin identities for spin-1 Ising system [72] which are given by

$$\langle \langle \{f_i\} s_i \rangle \rangle_r = \left\langle \left\langle \{f_i\} \frac{Tr_i s_i \exp(-\beta \mathcal{H}_i)}{Tr_i \exp(-\beta \mathcal{H}_i)} \right\rangle \right\rangle_r \quad (2)$$

$$\langle \langle \{f_i\} s_i^2 \rangle \rangle_r = \left\langle \left\langle \{f_i\} \frac{Tr_i s_i^2 \exp(-\beta \mathcal{H}_i)}{Tr_i \exp(-\beta \mathcal{H}_i)} \right\rangle \right\rangle_r \quad (3)$$

where Tr_i is the partial trace over the site i , $\beta = 1/(k_B T)$, k_B is Boltzmann constant and T is the temperature, $\{f_i\}$ is any function as long as it is not a function of the site i , \mathcal{H}_i is the part of the Hamiltonian which includes all contributions associated with the site i . While the inner average brackets stands for the thermal average, outer average brackets (with subscript r) stands for the random configurational average which is necessary for including random distribution effects.

Let us denote s_i^2 as q_i . Performing partial traces in Eqs. (2) and (3) with $\mathcal{H}_0 = -s_0 \sum_{\delta=1}^z J_{0\delta} s_\delta - q_0 D_0 - s_0 H_0$ for central spin (s_0) and with $\mathcal{H}_\delta = -s_\delta J_{0\delta} s_0 - s_\delta \gamma - q_\delta D_\delta - s_\delta H_\delta$ for perimeter spin (s_δ) then with using differential operator technique [23] will yield

$$\langle \langle \{f_0\} s_0 \rangle \rangle_r = \left\langle \left\langle \{f_0\} \prod_{\delta=1}^z [1 + s_\delta \sinh(J_{0\delta} \nabla) + q_\delta (\cosh(J_{0\delta} \nabla) - 1)] \right\rangle \right\rangle_r F_1(x)|_{x=0} \quad (4)$$

$$\langle \langle \{f_0\} q_0 \rangle \rangle_r = \left\langle \left\langle \{f_0\} \prod_{\delta=1}^z [1 + s_\delta \sinh(J_{0\delta} \nabla) + q_\delta (\cosh(J_{0\delta} \nabla) - 1)] \right\rangle \right\rangle_r F_2(x)|_{x=0} \quad (5)$$

$$\langle\langle\{f_\delta\}s_\delta\rangle\rangle_r = \langle\langle\{f_\delta\}[1 + s_0 \sinh(J_{0\delta}\nabla) + q_0 (\cosh(J_{0\delta}\nabla) - 1)]\rangle\rangle_r F_1(x + \gamma)|_{x=0} \quad (6)$$

$$\langle\langle\{f_\delta\}q_\delta\rangle\rangle_r = \langle\langle\{f_\delta\}[1 + s_0 \sinh(J_{0\delta}\nabla) + q_0 (\cosh(J_{0\delta}\nabla) - 1)]\rangle\rangle_r F_2(x + \gamma)|_{x=0} \quad (7)$$

where $\nabla = \partial/\partial x$ differential operator, $\gamma = (z - 1)h$ and h is the effective field per spin. The effect of the differential operator on a function is given by

$$\exp(a\nabla)F_j(x) = F_j(x + a), \quad j = 1, 2 \quad (8)$$

with any constant a .

Eqs. (4)-(7) are our fundamental equations and the functions in the equations are given by via,

$$G_1(x, D_i, H_i) = \frac{2 \sinh[\beta(x + H_i)]}{2 \cosh[\beta(x + H_i)] + \exp(-\beta D_i)}, \quad G_2(x, D_i, H_i) = \frac{2 \cosh[\beta(x + H_i)]}{2 \cosh[\beta(x + H_i)] + \exp(-\beta D_i)} \quad (9)$$

as

$$F_j(x) = \int dD_i dH_i P_D(D_i) P_H(H_i) G_j(x, D_i, H_i), \quad j = 1, 2 \quad (10)$$

where $P_D(D_i)$ and $P_H(H_i)$ are probability distributions of the crystal field and longitudinal magnetic field respectively.

We can obtain the central spin magnetization and the quadrupolar moment from Eqs. (4) and (5) by letting $\{f_0\} = 1$ and expanding right hand sides of them. Both expansions have multi site spin correlations like $\langle\langle s_1 s_2 \dots q_{k-1} q_k \rangle\rangle_r$ where $k \leq z$. The main problem here is to handle these correlations. These correlations are treated with DA in most of the works in literature due to the mathematical difficulties. In DA these correlations are treated according to

$$\langle\langle s_1 s_2 \dots q_{k-1} q_k \rangle\rangle_r = \langle\langle s_1 \rangle\rangle_r \langle\langle s_2 \rangle\rangle_r \dots \langle\langle q_{k-1} \rangle\rangle_r \langle\langle q_k \rangle\rangle_r \quad (11)$$

i.e neglecting all the multi site spin correlations. But these correlations can be obtained by choosing suitable form of $\{f_\delta\}$ in Eqs. (6) or (7). This is the main point of this work. If we choose a proper form of $\{f_\delta\}$ in Eqs. (6) or (7), this time other multi site correlations appear on the right of sides of Eqs. (6) and (7), which include the central site. By the same way, these can be obtained from Eqs. (4) or (5) with a suitable $\{f_0\}$. At the end of this process we can obtain a set of linear equations which includes these correlations as unknowns. For a given set of Hamiltonian parameters, we can obtain the effective field (which represents the effect of the outer $(z - 1)$ spins) by usual condition that

$$\langle\langle s_0 \rangle\rangle_r = \langle\langle s_\delta \rangle\rangle_r. \quad (12)$$

Once effective field h is determined for a given parameters, we can solve the system of linear equations and obtain the whole correlations of the system for given parameters. Since the effective field is very small in the vicinity of the phase transition point, the critical temperature can be obtained by letting $\gamma \rightarrow 0$.

In the formulation presented here, instead of deriving all correlations in the system, we construct representative correlations. Let us denote the correlation $\langle\langle s_1 s_2 \dots s_p q_{p+1} q_{p+2} \dots q_n \rangle\rangle_r$ as

$$\left\langle\left\langle s_\delta^{(p)} q_\delta^{(n-p)} \right\rangle\right\rangle_r = \langle\langle s_1 s_2 \dots s_p q_{p+1} q_{p+2} \dots q_n \rangle\rangle_r. \quad (13)$$

For instance, if we choose $n = 2, p = 1$ in Eq. (13) for the system with $z = 3$, the correlation $\left\langle\left\langle s_\delta^{(1)} q_\delta^{(1)} \right\rangle\right\rangle_r = \langle\langle s_1 q_2 \rangle\rangle_r$ represents all the terms $\langle\langle s_1 q_2 \rangle\rangle_r, \langle\langle s_1 q_3 \rangle\rangle_r, \langle\langle s_2 q_1 \rangle\rangle_r, \langle\langle s_2 q_3 \rangle\rangle_r, \langle\langle s_3 q_1 \rangle\rangle_r, \langle\langle s_3 q_2 \rangle\rangle_r$. This means that our solutions are under the assumption of that all these correlations are equal to each other. In general, we make an assumption that all n perimeter site correlations which includes p number of s_δ and $n - p$ number of q_δ are equal to each other and we represent them by $\left\langle\left\langle s_\delta^{(p)} q_\delta^{(n-p)} \right\rangle\right\rangle_r$ as given in Eq. (13). The same assumption holds for the correlations which include central site and n perimeter sites, i.e all correlations which include s_0 and p number of s_δ and $n - p$ number of q_δ are equal to each other and all correlations which

include q_0 and p number of s_δ and $n - p$ number of q_δ are equal to each other, and we represent them as $\langle\langle s_0 s_\delta^{(p)} q_\delta^{(n-p)} \rangle\rangle_r$ and $\langle\langle q_0 s_\delta^{(p)} q_\delta^{(n-p)} \rangle\rangle_r$, respectively.

Deriving correlations simply based on choosing $\{f_0\}$ in Eqs. (4) or (5), or $\{f_\delta\}$ in Eqs. (6) or (7) and chosen $\{f_0\}$ and $\{f_\delta\}$, will change the operator in the average brackets on the right hand sides of them. For example, by the choice of $\{f_0\} = s_k$ in Eq. (4) one applies s_k to the expression $\langle\langle s_0 \rangle\rangle_r$. Applying s_1 to the expression $\langle\langle s_0 \rangle\rangle_r$ will create the correlation $\langle\langle s_0 s_1 \rangle\rangle_r$. After then applying s_2 to the $\langle\langle s_0 s_1 \rangle\rangle_r$ will create $\langle\langle s_0 s_1 s_2 \rangle\rangle_r$ correlation, and so on. Thus, we get the correlation $\langle\langle s_0 s_1 \dots s_z \rangle\rangle_r$ with successive applications of s_k to the correlation $\langle\langle s_0 s_1 \dots s_{k-1} \rangle\rangle_r$ ($k = 1, 2, \dots, z$) in z steps.

Since our starting points of the derivation process are Eqs. (4)-(7) we can write any correlation in terms of the related operator which is applied to the related function, as in expressions Eqs. (4)-(7). Since the operators on the right hand sides of Eqs. (4) and (5) are the same, we will use the same operator for getting the correlations e.g. $\langle\langle s_0 s_1 s_2 \rangle\rangle_r$ and $\langle\langle q_0 s_1 s_2 \rangle\rangle_r$. The same reasoning also holds for the derivation of the correlations from Eqs. (6) and (7). Thus, instead of deriving all correlations separately in sequential orders, we will concentrate on the evolution of these operators in that sequential orders. Let us denote the operator on the right hand sides of Eqs. (4) and (5) for $\{f_0\} = 1$ as $\Theta_{0,0}$ and the operator on the right hand sides of Eqs. (6) and (7) with $\delta = 1, \{f_1\} = 1$ as $\Phi_{0,0}$. With these operators we can write our fundamental equations given in Eqs. (4)-(7) as follows:

$$\langle\langle s_0 \rangle\rangle_r = \langle\langle \Theta_{00} \rangle\rangle_r F_1(x)|_{x=0} \quad (14)$$

$$\langle\langle s_1 \rangle\rangle_r = \langle\langle \Phi_{00} \rangle\rangle_r F_1(x + \gamma)|_{x=0} \quad (15)$$

$$\langle\langle q_0 \rangle\rangle_r = \langle\langle \Theta_{00} \rangle\rangle_r F_2(x)|_{x=0} \quad (16)$$

$$\langle\langle q_1 \rangle\rangle_r = \langle\langle \Phi_{00} \rangle\rangle_r F_2(x + \gamma)|_{x=0} \quad (17)$$

where the operators are given as

$$\Theta_{00} = \prod_{\delta=1}^z [1 + s_\delta \sinh(J_{0\delta} \nabla) + q_\delta (\cosh(J_{0\delta} \nabla) - 1)] \quad (18)$$

$$\Phi_{00} = [1 + s_0 \sinh(J_{0\delta} \nabla) + q_0 (\cosh(J_{0\delta} \nabla) - 1)]. \quad (19)$$

Let us derive correlations in sequential orders given below. At each step of the derivation, the operators evolve in the sequences as given below:

$$\begin{array}{ccccccc} \langle\langle s_1 \rangle\rangle_r & \rightarrow & \langle\langle s_1 s_2 \rangle\rangle_r & \rightarrow & \dots & \rightarrow & \langle\langle s_1 s_2 \dots s_z \rangle\rangle_r \\ \Phi_{00} & \rightarrow & \Phi_{10} & \rightarrow & \dots & \rightarrow & \Phi_{z-1,0} \end{array} \quad (20)$$

$$\begin{array}{ccccccc} \langle\langle s_0 \rangle\rangle_r & \rightarrow & \langle\langle s_0 s_1 \rangle\rangle_r & \rightarrow & \langle\langle s_0 s_1 s_2 \rangle\rangle_r & \rightarrow & \dots \rightarrow \langle\langle s_0 s_1 s_2 \dots s_z \rangle\rangle_r \\ \Theta_{00} & \rightarrow & \Theta_{10} & \rightarrow & \Theta_{20} & \rightarrow & \dots \rightarrow \Theta_{z0} \end{array} \quad (21)$$

$$\begin{array}{ccccccc} \langle\langle s_1 s_2 \dots s_{k+1} \rangle\rangle_r & \rightarrow & \langle\langle s_1 s_2 \dots s_k q_{k+1} \rangle\rangle_r & \rightarrow & \dots & \rightarrow & \langle\langle s_1 s_2 \dots s_{k-m+1} q_{k-m+2} \dots q_{k+1} \rangle\rangle_r \\ \Phi_{k0} & \rightarrow & \Phi_{k1} & \rightarrow & \dots & \rightarrow & \Phi_{km} \end{array} \quad (22)$$

$$\begin{array}{ccccccc} \langle\langle s_0 s_1 s_2 \dots s_k \rangle\rangle_r & \rightarrow & \langle\langle s_0 s_1 s_2 \dots s_{k-1} q_k \rangle\rangle_r & \rightarrow & \dots & \rightarrow & \langle\langle s_0 s_1 s_2 \dots s_{k-m} q_{k-m+1} \dots q_k \rangle\rangle_r \\ \Theta_{k0} & \rightarrow & \Theta_{k1} & \rightarrow & \dots & \rightarrow & \Theta_{km} \end{array} \quad (23)$$

In Eqs. (22) and (23) we use the self spin identities of spin-1 Ising system during the derivation process, which are given as follows.

$$s_i q_i = s_i, s_i s_i = q_i, q_i q_i = q_i, \quad i = 0, 1, \dots, z. \quad (24)$$

According to Eq. (24), applying s_l to the correlation $\langle\langle s_1 s_2 \dots s_l q_{l+1} \dots q_k \rangle\rangle_r$ will produce the correlation $\langle\langle s_1 s_2 \dots s_{l-1} q_l q_{l+1} \dots q_k \rangle\rangle_r$ where $l \leq k$, which can be obtained by applying the $\langle\langle \Phi_{k-1, k-l+1} \rangle\rangle_r$ operator to the function $F_1(x + \gamma)$ and taking the value of resultant expression at $x = 0$.

Sequential orders given in Eqs. (20)-(23), expose the recurrence relations for the operators as:

$$\begin{aligned}
\Phi_{k,0} &= s_{k+1}\Phi_{k-1,0}, & k &= 1, 2, \dots, z-1 \\
\Theta_{k,0} &= s_k\Theta_{k-1,0}, & k &= 1, 2, \dots, z \\
\Phi_{k,m} &= s_{k-m+2}\Phi_{k,m-1}, & m &= 1, 2, \dots, k \\
\Theta_{k,m} &= s_{k-m+1}\Theta_{k,m-1}, & m &= 1, 2, \dots, k.
\end{aligned} \tag{25}$$

In order to obtain these operators let us start with the operator Θ_{00} which is just the operators on the right hand sides of Eqs. (4) and (5) with $\{f_0\} = 1$. By using the assumption about the correlations which is explained above (under the Eq. (13)), we can write this operator as

$$\Theta_{00} = \sum_{n=0}^z \sum_{p=0}^n A_{np} s_{\delta}^{(p)} q_{\delta}^{(n-p)} \tag{26}$$

where

$$A_{np} = \binom{z}{n} \binom{n}{p} \prod_{\delta=1}^p \sinh(J_{0\delta}\nabla) \prod_{\delta=p+1}^n (\cosh(J_{0\delta}\nabla) - 1). \tag{27}$$

However, obtaining Θ_{k0} from $\Theta_{k-1,0}$ by using Eq. (25) requires the determination of any term $s_k s_{\delta}^{(l)} q_{\delta}^{(m)}$ which will appear on the right hand side of Θ_{k0} . This requirement is also valid for any process for obtaining the operators by using the recurrence relations given in Eq. (25). At this stage we use self spin correlations for the spin-1 Ising system given in Eq. (24). Hence there exist three possibilities for the term $s_k s_{\delta}^{(l)} q_{\delta}^{(m)}$ according to Eq. (24):

$$s_k s_{\delta}^{(l)} q_{\delta}^{(m)} = \begin{cases} s_{\delta}^{(l+1)} q_{\delta}^{(m)} & , \quad k > l + m \\ s_{\delta}^{(l+1)} q_{\delta}^{(m-1)} & , \quad l < k \leq l + m \\ s_{\delta}^{(l-1)} q_{\delta}^{(m+1)} & , \quad 0 < k \leq l \end{cases} \tag{28}$$

Now, one strategy is to obtain Θ_{k0} from Eq. (25) starting by Eq. (26) is making first few iterations and trying to capture the general expression for the Θ_{k0} . However we follow a slightly different strategy here. Each of the iterations in Eq. (25) takes the coefficient A_{np} which is the multiplier of some $s_{\delta}^{(l)} q_{\delta}^{(m)}$ and place it to another term $s_{\delta}^{(l')} q_{\delta}^{(m')}$ as a multiplier. We envision that the iterations in Eq. (25) will generate certain movement of the coefficients A_{np} between the terms $s_{\delta}^{(l)} q_{\delta}^{(m)}$ as a multiplier of them. Then if we can track every coefficient during the each step of the iterations, we can determine the place of any coefficient for that iteration step. Thus we can write an expression for any operator in any iteration step.

For instance let us take the coefficient A_{00} and track this coefficient during the derivation in a sequential order given in Eq. (21).

$$A_{00} s_{\delta}^{(0)} q_{\delta}^{(0)} \rightarrow A_{00} s_{\delta}^{(1)} q_{\delta}^{(0)} \rightarrow \dots \rightarrow A_{00} s_{\delta}^{(k)} q_{\delta}^{(0)} \rightarrow \dots \rightarrow A_{00} s_{\delta}^{(z)} q_{\delta}^{(0)} \tag{29}$$

Eq. (29) implies that, the coefficient A_{00} is placed as a multiplier of the term $s_{\delta}^{(k)} q_{\delta}^{(0)}$ on the k .th step of the derivation, i.e the operator expression Θ_{k0} has the term $A_{00} s_{\delta}^{(k)} q_{\delta}^{(0)} = A_{00} s_1 s_2 \dots s_k$ on the right hand side. If we know the locations of the all A_{np} coefficients at the k .th step of the derivation we can get an expression for Θ_{k0} . This is the general strategy which is used to obtain all the operators in Eq. (21).

By inspection and taking into account Eq. (28), in order to get an expression for the operator Θ_{k0} , we can give the locations of the coefficients of the k .th step of the derivation in an order given in Eq. (21) as follows.

$$A_{00} \rightarrow s_{\delta}^{(k)} q_{\delta}^{(0)}. \tag{30}$$

For $n > 0$

$$A_{n0} \rightarrow \begin{cases} s_{\delta}^{(k)} q_{\delta}^{(n-k)} & , \quad k \leq n \\ s_{\delta}^{(k)} q_{\delta}^{(0)} & , \quad k > n \end{cases} \tag{31}$$

For $p \neq 0$ and even

$$A_{np} \rightarrow \begin{cases} s_{\delta}^{(p-k)} q_{\delta}^{(n-p+k)} & , \quad k \leq p/2 \\ s_{\delta}^{(k)} q_{\delta}^{(n-k)} & , \quad p/2 < k \leq n \\ s_{\delta}^{(k)} q_{\delta}^{(0)} & , \quad k > n \end{cases} \quad (32)$$

and for odd p

$$A_{np} \rightarrow \begin{cases} s_{\delta}^{(p-k)} q_{\delta}^{(n-p+k)} & , \quad k \leq (p+1)/2 \\ s_{\delta}^{(k-1)} q_{\delta}^{(n-k+1)} & , \quad (p+1)/2 < k \leq n \\ s_{\delta}^{(k-1)} q_{\delta}^{(1)} & , \quad k > n \end{cases} \quad (33)$$

Detailed derivation of Eqs. (30)-(33) is given in Section 6. From Eqs. (30)-(33) we can get an expression for the operator Θ_{k0} as

$$\begin{aligned} \Theta_{k,0} = & \left(D_k^{(1)} + D_k^{(2)} \right) s_{\delta}^{(k)} q_{\delta}^{(0)} + D_k^{(3)} s_{\delta}^{(k-1)} q_{\delta}^{(1)} + \sum_{n=k}^z \left(A_{n0} + D_{nk}^{(4)} \right) s_{\delta}^{(k)} q_{\delta}^{(n-k)} \\ & + \sum_{n=k}^z D_{nk}^{(5)} s_{\delta}^{(k-1)} q_{\delta}^{(n-k+1)} + \sum_{n=k, p=2k-1}^z \sum_{n=1}^n A_{np} s_{\delta}^{(p-k)} q_{\delta}^{(n-p+k)} \end{aligned} \quad (34)$$

where $k = 1, 2, \dots, z$ and the coefficients are given by

$$\begin{aligned} D_k^{(1)} &= \sum_{n=0}^{k-1} A_{n0}, \quad D_k^{(2)} = \sum_{n=1}^{k-1} \sum_{p=2}^{n''} A_{np}, \quad D_k^{(3)} = \sum_{n=1}^{k-1} \sum_{p=1}^{n'} A_{np} \\ D_{nk}^{(4)} &= \sum_{p=2}^{2k-1''} A_{np}, \quad D_{nk}^{(5)} = \sum_{p=1}^{2k-2'} A_{np}. \end{aligned} \quad (35)$$

The $'$ sign on the upper limits of the sums in (35) indicate that, sum runs over the odd indexes while the $''$ sign on the upper limits of the sums indicate that sum runs over the even indexes.

In the sequential order given in Eq. (23), we can obtain the operator Θ_{km} after a few iterations by using Eq. (28), and it is given by

$$\begin{aligned} \Theta_{k,m} = & \left(D_k^{(1)} + D_k^{(2)} \right) s_{\delta}^{(k-m)} q_{\delta}^{(m)} + D_k^{(3)} s_{\delta}^{(k-m+1)} q_{\delta}^{(m-1)} \\ & + \sum_{n=k}^z \left(A_{n0} + D_{nk}^{(4)} \right) s_{\delta}^{(k-m)} q_{\delta}^{(n-k+m)} \\ & + \sum_{n=k}^z \left(A_{n,2k-1} + D_{nk}^{(5)} \right) s_{\delta}^{(k-m+1)} q_{\delta}^{(n-k+m-1)} + \sum_{n=k, p=2k}^z \sum_{n=1}^n A_{np} s_{\delta}^{(p-k-m)} q_{\delta}^{(n-p+k+m)} \end{aligned} \quad (36)$$

where $k = 1, 2, \dots, z$; $m = 1, 2, \dots, k$.

From Eqs. (6) and (7) we can see the form of the operator Φ_{00} defined in Eqs. (15) and (17) as:

$$\Phi_{00} = B_0 + B_1 s_0 + B_2 q_0 \quad (37)$$

where

$$\begin{aligned} B_0 &= 1 \\ B_1 &= \sinh(J_{0\delta} \nabla) \\ B_2 &= \cosh(J_{0\delta} \nabla) - 1. \end{aligned} \quad (38)$$

If we use the first line of the Eq. (25) by taking into account Eq. (28), we can obtain the operator Φ_{k0} after a few iterations as,

$$\Phi_{k0} = B_0 s_{\delta}^{(k)} q_{\delta}^{(0)} + B_1 s_0 s_{\delta}^{(k)} q_{\delta}^{(0)} + B_2 q_0 s_{\delta}^{(k)} q_{\delta}^{(0)} \quad (39)$$

where $k = 0, 1, \dots, z-1$. After then, by the same way, using the third line of the Eq. (25) we get

$$\Phi_{km} = B_0 s_{\delta}^{(k-m)} q_{\delta}^{(m)} + B_1 s_0 s_{\delta}^{(k-m)} q_{\delta}^{(m)} + B_2 q_0 s_{\delta}^{(k-m)} q_{\delta}^{(m)} \quad (40)$$

where $k = 0, 1, \dots, z-1$ and $m = 0, 1, \dots, k$. By using Eqs. (24) and (40) we can obtain the operator $s_0\Phi_{km}$ and $q_0\Phi_{km}$ as

$$\begin{aligned} s_0\Phi_{km} &= (B_0 + B_2)s_0s_\delta^{(k-m)}q_\delta^{(m)} + B_1q_0s_\delta^{(k-m)}q_\delta^{(m)} \\ q_0\Phi_{km} &= (B_0 + B_2)q_0s_\delta^{(k-m)}q_\delta^{(m)} + B_1s_0s_\delta^{(k-m)}q_\delta^{(m)} \end{aligned} \quad (41)$$

where $k = 0, 1, \dots, z-1$, $m = 0, 1, \dots, k$ and this completes the process of derivation of operators. Note that the definition given in Eq. (40) covers the definition given in Eq. (39) with $m = 0$ in it, but the same thing does not hold Eq. (36) for Eq. (34) i.e the operator $\Theta_{k,0}$ have to be calculated from Eq. (34).

Now, since the operators and functions are well defined, we can construct the correlations. But there are more than one way to constructing any correlation. Thus we have to determine an appropriate expressions from the equalities below, for deriving correlations.

The number of $z/2(z+3) - 2$ correlations of the type $\left\langle\left\langle s_\delta^{(k-m)}q_\delta^{(m)} \right\rangle\right\rangle_r$ ($k = 2, 3, \dots, z$ and $m = 0, 1, \dots, k$) can be obtained by starting from Eq. (15) or (17) as

$$\left\langle\left\langle s_\delta^{(k-m)}q_\delta^{(m)} \right\rangle\right\rangle_r = \langle\langle\Phi_{k-1,m}\rangle\rangle_r F_1(x+\gamma)|_{x=0}, m = 0, 1, \dots, k-1 \quad (42)$$

$$\left\langle\left\langle s_\delta^{(k-m)}q_\delta^{(m)} \right\rangle\right\rangle_r = \langle\langle\Phi_{k-1,m-1}\rangle\rangle_r F_2(x+\gamma)|_{x=0}, m = 1, 2, \dots, k \quad (43)$$

respectively, where $k = 2, 3, \dots, z$. In order to obtain the complete set of correlations of this type, we can not use only one of Eqs. (42) and (43) due to the restrictions on m values of them.

Since the operator $\Theta_{k,m}$ defined in Eq. (36) can not have $m = 0$ value, the correlations of the type $\left\langle\left\langle s_0s_\delta^{(k)} \right\rangle\right\rangle_r$ are obtained separately from the correlations of the type $\left\langle\left\langle s_0s_\delta^{(k-m)}q_\delta^{(m)} \right\rangle\right\rangle_r$ ($m \neq 0$). z number of correlations of the type $\left\langle\left\langle s_0s_\delta^{(k)} \right\rangle\right\rangle_r$ ($k = 1, 2, \dots, z$) can be obtained by starting from Eqs. (14) or (15) and they are given respectively as

$$\left\langle\left\langle s_0s_\delta^{(k)} \right\rangle\right\rangle_r = \langle\langle\Theta_{k,0}\rangle\rangle_r F_1(x)|_{x=0} \quad (44)$$

$$\left\langle\left\langle s_0s_\delta^{(k)} \right\rangle\right\rangle_r = \langle\langle s_0\Phi_{k-1,0}\rangle\rangle_r F_1(x+\gamma)|_{x=0}. \quad (45)$$

$z/2(z+1)$ number of correlations of the type $\left\langle\left\langle s_0s_\delta^{(k-m)}q_\delta^{(m)} \right\rangle\right\rangle_r$ ($k = 1, 2, \dots, z$ and $m = 1, 2, \dots, k$) can be obtained by starting from Eqs. (14) or (17) and they are given as

$$\left\langle\left\langle s_0s_\delta^{(k-m)}q_\delta^{(m)} \right\rangle\right\rangle_r = \langle\langle\Theta_{k,m}\rangle\rangle_r F_1(x)|_{x=0} \quad (46)$$

$$\left\langle\left\langle s_0s_\delta^{(k-m)}q_\delta^{(m)} \right\rangle\right\rangle_r = \langle\langle s_0\Phi_{k-1,m-1}\rangle\rangle_r F_2(x+\gamma)|_{x=0} \quad (47)$$

respectively.

In order to get the complete set of correlations of the type $\left\langle\left\langle s_0s_\delta^{(k-m)}q_\delta^{(m)} \right\rangle\right\rangle_r$ including $m = 0$ we can also use

$$\left\langle\left\langle s_0s_\delta^{(k-m)}q_\delta^{(m)} \right\rangle\right\rangle_r = \langle\langle s_0\Phi_{k-1,m}\rangle\rangle_r F_1(x+\gamma)|_{x=0}, m = 0, 1, \dots, k-1 \quad (48)$$

with Eq. (47). For $k = 1, 2, \dots, z$, either Eq. (48) with $m = 0$ and Eq. (47) with $m = 1, 2, \dots, k$ or Eq. (47) with $m = k$ and Eq. (48) with $m = 0, 1, \dots, k-1$ will generate number of $z/2(z+3)$ correlations which are the complete set of correlations including s_0 .

Number of z correlations of the type $\left\langle\left\langle q_0s_\delta^{(k)} \right\rangle\right\rangle_r$ ($k = 1, 2, \dots, z$) can be obtained by starting from Eqs. (15) or (16) and they are given respectively as

$$\left\langle\left\langle q_0s_\delta^{(k)} \right\rangle\right\rangle_r = \langle\langle q_0\Phi_{k-1,0}\rangle\rangle_r F_1(x+\gamma)|_{x=0} \quad (49)$$

$$\left\langle \left\langle q_0 s_\delta^{(k)} \right\rangle \right\rangle_r = \langle \langle \Theta_{k,0} \rangle \rangle_r F_2(x)|_{x=0} \quad (50)$$

Number of $z/2(z+1)$ correlations of the type $\left\langle \left\langle q_0 s_\delta^{(k-m)} q_\delta^{(m)} \right\rangle \right\rangle_r$ ($k = 1, 2, \dots, z$ and $m = 1, 2, \dots, k$) can be obtained by starting from Eqs. (16) or (17) and they are given as

$$\left\langle \left\langle q_0 s_\delta^{(k-m)} q_\delta^{(m)} \right\rangle \right\rangle_r = \langle \langle \Theta_{k,m} \rangle \rangle_r F_2(x)|_{x=0} \quad (51)$$

$$\left\langle \left\langle q_0 s_\delta^{(k-m)} q_\delta^{(m)} \right\rangle \right\rangle_r = \langle \langle q_0 \Phi_{k-1,m-1} \rangle \rangle_r F_2(x+\gamma)|_{x=0} \quad (52)$$

respectively. Again, the complete set of correlations of the type $\left\langle \left\langle q_0 s_\delta^{(k-m)} q_\delta^{(m)} \right\rangle \right\rangle_r$ including $m = 0$ can be obtained from

$$\left\langle \left\langle q_0 s_\delta^{(k-m)} q_\delta^{(m)} \right\rangle \right\rangle_r = \langle \langle q_0 \Phi_{k-1,m} \rangle \rangle_r F_1(x+\gamma)|_{x=0}, m = 0, 1, \dots, k-1 \quad (53)$$

with Eq. (52). Due to the restrictions of m in Eq. (52) and Eq. (53), we can obtain the complete set of correlations which include q_0 , from Eq. (53) with $m = 0$ and (52) with $m = 1, 2, \dots, k$ or (52) with $m = k$ and (53) with $m = 0, 1, \dots, k-1$.

It is possible to obtain number of $3z/2(z+3) - 2$ correlations from Eqs. (42)-(53) and then to construct the system of linear equations (which has correlations and fundamental equalities given in (14)-(17), as equations) which has the number of $3z/2(z+3) + 2$ equations. However, in order to ensure the correctness of the correlation expressions given in Eqs. (42)-(53) we have to investigate the spin-1/2 limits of them. It is a well known fact that the spin-1 system behaves like spin-1/2 system for large positive values of the crystal field i.e. in the limit of $D \rightarrow \infty$. This means that in this limit, $s_i = 0$ becomes as an inaccessible state for the spins on the lattice sites. As a result of this, in this limit, all q_i terms in expressions become 1. This criteria impose that Eq. (46) can not be used for deriving that correlations. This fact is explained in Section 7 in detail.

From Eqs. (14)-(17) and the definition of the related operators given in Eqs. (26) and (37), we can obtain the fundamental equations of the central and perimeter spin magnetizations and quadrupolar moments as

$$\langle \langle s_0 \rangle \rangle_r = \sum_{n=0}^z \sum_{p=0}^n C_{np}^{(1)} \left\langle \left\langle s_\delta^{(p)} q_\delta^{(n-p)} \right\rangle \right\rangle_r \quad (54)$$

$$\langle \langle q_0 \rangle \rangle_r = \sum_{n=0}^z \sum_{p=0}^n C_{np}^{(2)} \left\langle \left\langle s_\delta^{(p)} q_\delta^{(n-p)} \right\rangle \right\rangle_r \quad (55)$$

$$\langle \langle s_1 \rangle \rangle_r = K_0^{(1)} + K_1^{(1)} \langle \langle s_0 \rangle \rangle_r + K_2^{(1)} \langle \langle q_0 \rangle \rangle_r \quad (56)$$

$$\langle \langle q_1 \rangle \rangle_r = K_0^{(2)} + K_1^{(2)} \langle \langle s_0 \rangle \rangle_r + K_2^{(2)} \langle \langle q_0 \rangle \rangle_r \quad (57)$$

where ($i = 1, 2$ and $j = 0, 1, 2$)

$$\begin{aligned} C_{np}^{(i)} &= \langle A_{np} \rangle_r F_i(x)|_{x=0} \\ K_j^{(i)} &= \langle B_j \rangle_r F_i(x+\gamma)|_{x=0}. \end{aligned} \quad (58)$$

Configurational averages in Eq. (58) can be evaluated by the given probability distribution of bonds $P_J(J_{ij})$ and the functions $F_i(x)$ are given by Eq. (10), coefficients A_{np} and B_j are given by Eqs. (27) and (38), respectively.

We can obtain the correlations which has only perimeter sites, by using Eq. (40) in Eqs. (42) and (43) with $k = 2, 3, \dots, z$. From Eq. (42) with $m = 0$ we have

$$\left\langle \left\langle s_\delta^{(k)} q_\delta^{(0)} \right\rangle \right\rangle_r = K_0^{(1)} \left\langle \left\langle s_\delta^{(k-1)} q_\delta^{(0)} \right\rangle \right\rangle_r + K_1^{(1)} \left\langle \left\langle s_0 s_\delta^{(k-1)} q_\delta^{(0)} \right\rangle \right\rangle_r + K_2^{(1)} \left\langle \left\langle q_0 s_\delta^{(k-1)} q_\delta^{(0)} \right\rangle \right\rangle_r \quad (59)$$

and from Eq. (43) with $m = 1, 2, \dots, k$ we obtain

$$\left\langle \left\langle s_\delta^{(k-m)} q_\delta^{(m)} \right\rangle \right\rangle_r = K_0^{(2)} \left\langle \left\langle s_\delta^{(k-m)} q_\delta^{(m-1)} \right\rangle \right\rangle_r + K_1^{(2)} \left\langle \left\langle s_0 s_\delta^{(k-m)} q_\delta^{(m-1)} \right\rangle \right\rangle_r + K_2^{(2)} \left\langle \left\langle q_0 s_\delta^{(k-m)} q_\delta^{(m-1)} \right\rangle \right\rangle_r. \quad (60)$$

For correlations that include s_0 and perimeter sites, we can use operator expressions Eqs. (34) and (41), in Eqs. (44) and (47) respectively with $k = 1, 2, \dots, z$. From Eq. (44) we get

$$\begin{aligned} \langle \langle s_0 s_\delta^{(k)} q_\delta^{(0)} \rangle \rangle_r &= \left(L_k^{(1,1)} + L_k^{(1,2)} \right) \langle \langle s_\delta^{(k)} q_\delta^{(0)} \rangle \rangle_r + L_k^{(1,3)} \langle \langle s_\delta^{(k-1)} q_\delta^{(1)} \rangle \rangle_r + \sum_{n=k}^z \left(C_{n0}^{(1)} + L_{nk}^{(1,4)} \right) \langle \langle s_\delta^{(k)} q_\delta^{(n-k)} \rangle \rangle_r \\ &+ \sum_{n=k}^z L_{nk}^{(1,5)} \langle \langle s_\delta^{(k-1)} q_\delta^{(n-k+1)} \rangle \rangle_r + \sum_{n=k, p=2k-1}^z \sum_{n=2k-1}^n C_{np}^{(1)} \langle \langle s_\delta^{(p-k)} q_\delta^{(n-p+k)} \rangle \rangle_r \end{aligned} \quad (61)$$

where $k = 1, 2, \dots, z$ and from Eq. (47) with $m = 1, 2, \dots, k$

$$\langle \langle s_0 s_\delta^{(k-m)} q_\delta^{(m)} \rangle \rangle_r = (K_0^{(2)} + K_2^{(2)}) \langle \langle s_0 s_\delta^{(k-m)} q_\delta^{(m-1)} \rangle \rangle_r + K_1^{(2)} \langle \langle q_0 s_\delta^{(k-m)} q_\delta^{(m-1)} \rangle \rangle_r \quad (62)$$

Similarly, for the correlations that include q_0 and perimeter sites, we can use the operator expressions given in Eqs. (34) and (36), in Eqs. (50) and (51), respectively with $k = 1, 2, \dots, z$. From Eq. (50)

$$\begin{aligned} \langle \langle q_0 s_\delta^{(k)} q_\delta^{(0)} \rangle \rangle_r &= \left(L_k^{(2,1)} + L_k^{(2,2)} \right) \langle \langle s_\delta^{(k)} q_\delta^{(0)} \rangle \rangle_r + L_k^{(2,3)} \langle \langle s_\delta^{(k-1)} q_\delta^{(1)} \rangle \rangle_r + \sum_{n=k}^z \left(C_{n0}^{(2)} + L_{nk}^{(2,4)} \right) \langle \langle s_\delta^{(k)} q_\delta^{(n-k)} \rangle \rangle_r \\ &+ \sum_{n=k}^z L_{nk}^{(2,5)} \langle \langle s_\delta^{(k-1)} q_\delta^{(n-k+1)} \rangle \rangle_r + \sum_{n=k, p=2k-1}^z \sum_{n=2k-1}^n C_{np}^{(2)} \langle \langle s_\delta^{(p-k)} q_\delta^{(n-p+k)} \rangle \rangle_r \end{aligned} \quad (63)$$

and from Eq. (51) with $m = 1, 2, \dots, k$

$$\begin{aligned} \langle \langle q_0 s_\delta^{(k-m)} q_\delta^{(m)} \rangle \rangle_r &= \left(L_k^{(2,1)} + L_k^{(2,2)} \right) \langle \langle s_\delta^{(k-m)} q_\delta^{(m)} \rangle \rangle_r + L_k^{(2,3)} \langle \langle s_\delta^{(k-m+1)} q_\delta^{(m-1)} \rangle \rangle_r \\ &+ \sum_{n=k}^z \left(C_{n0}^{(2)} + L_{nk}^{(2,4)} \right) \langle \langle s_\delta^{(k-m)} q_\delta^{(n-k+m)} \rangle \rangle_r \\ &+ \sum_{n=k}^z \left(C_{n, 2k-1}^{(2)} + L_{nk}^{(2,5)} \right) \langle \langle s_\delta^{(k-m+1)} q_\delta^{(n-k+m-1)} \rangle \rangle_r + \sum_{n=k, p=2k}^z \sum_{n=2k}^n C_{np}^{(2)} \langle \langle s_\delta^{(p-k-m)} q_\delta^{(n-p+k+m)} \rangle \rangle_r \end{aligned} \quad (64)$$

where the coefficients $L_k^{(ij)}$ and $L_{nk}^{(ij)}$ are given by ($i = 1, 2$)

$$\begin{aligned} L_k^{(ij)} &= \left\langle D_k^{(j)} \right\rangle_r F_i(x)|_{x=0}, \quad j = 1, 2, 3 \\ L_{nk}^{(ij)} &= \left\langle D_{nk}^{(j)} \right\rangle_r F_i(x)|_{x=0}, \quad j = 4, 5 \end{aligned} \quad (65)$$

and this completes the derivation of correlations process.

Fundamental equalities in Eqs. (54)-(57) and Eqs. (59)-(64) constitute linear equation system which includes number of $3/2z(z+3)+2$ equations of correlations for the system with coordination number z . By appropriately indexing the correlations, this system of equations can be written in a matrix form and can be solved numerically. If the system is defined by coordination number z with bond, crystal field and longitudinal magnetic field probability distributions ($P_J(J_{ij})$, $P_D(D_i)$ and $P_H(H_i)$ respectively), then the functions in coefficients can be calculated from Eqs. (9), (10), the coefficients in the operators from Eqs. (27), (35), (38) and finally the coefficients in the correlation equalities from Eqs. (58), (65) with using this distribution functions.

Instead of deriving correlations by using Eqs. (42), (43), (44), (47), (50) and (51); using Eqs. (42), (43), (47), (48), (52) and (53) for correlations and Eqs. (54)-(57) for fundamental equalities will generate another set of linear equations which gives results that are identical to those obtained in [23], which is just the extended BPA of the spin-1/2 system to the spin-1 system.

For completeness of our work, we give the DA formulation corresponding to Eqs. (54) and (55) with the help of Eq. (11) as

$$\langle \langle s_0 \rangle \rangle_r = \sum_{n=0}^z \sum_{p=0}^n C_{np}^{(1)} \langle \langle s_0 \rangle \rangle_r^p \langle \langle q_0 \rangle \rangle_r^{n-p} \quad (66)$$

$$\langle\langle q_0 \rangle\rangle_r = \sum_{n=0}^z \sum_{p=0}^n C_{np}^{(2)} \langle\langle s_0 \rangle\rangle_r^p \langle\langle q_0 \rangle\rangle_r^{n-p} \quad (67)$$

By solving the system of nonlinear equations given by Eqs. (66), (67) with the coefficients in (58) we get DA results.

4 Results and Discussion

Since the aim of our work is to develop a general EFT formulation which is capable of calculating multi site spin correlations, we give only few results in comparison with DA and EFT approximations in the literature. Unless otherwise stated, we mean by DA results that the results obtained by the formulation given in Ref. [22] and the EFT results that obtained by the formulation given in Ref. [23]. The results of bond dilution problem on a honeycomb lattice ($z = 3$) and crystal field dilution problem can be found in our earlier works [74, 75], respectively.

4.1 Pure System With Zero Magnetic Field

In this simplest spin-1 system, all probability distribution functions can be given by delta functions as $P_J(J_{ij}) = \delta(J_{ij} - J)$, $P_D(D_i) = \delta(D_i - D)$ and $P_H(H_i) = \delta(H_i)$. The functions of this system can be calculated from Eqs. (9) and (10) by using these probability distribution functions for crystal field and magnetic field and they are given as

$$F_1(x) = \frac{2 \sinh(\beta x)}{2 \cosh(\beta x) + \exp(-\beta D)}, \quad F_2(x) = \frac{2 \cosh(\beta x)}{2 \cosh(\beta x) + \exp(-\beta D)} \quad (68)$$

Since $J_{ij} = J$ for all bonds then (58) can be written as

$$C_{np}^{(k)} = \binom{z}{n} \binom{n}{p} [\sinh(J\nabla)]^p [\cosh(J\nabla) - 1]^{n-p} F_k(x)|_{x=0} \quad (69)$$

where $k = 1, 2$. We can write the hyperbolic operators in terms of the exponential operators in Eq. (69) and by making binomial expansion of them we can obtain:

$$C_{np}^{(k)} = \binom{z}{n} \binom{n}{p} \sum_{r=0}^{n-p} \sum_{i=0}^r \sum_{j=0}^p \binom{n-p}{r} \binom{r}{i} \binom{p}{j} \frac{1}{2^{r+p}} (-1)^{n-p-r+j} \exp[(r+p-2i-2j)J\nabla] F_k(x)|_{x=0} \quad (70)$$

if we apply the differential operator to the function $F_k(x)$ according to Eq. (8) we can get

$$C_{np}^{(k)} = \binom{z}{n} \binom{n}{p} \sum_{r=0}^{n-p} \sum_{i=0}^r \sum_{j=0}^p \binom{n-p}{r} \binom{r}{i} \binom{p}{j} \frac{1}{2^{r+p}} (-1)^{n-p-r+j} F_k[(r+p-2i-2j)J] \quad (71)$$

Other coefficients for this case can be obtained by the same way from Eq. (58) as

$$\begin{aligned} K_0^{(k)} &= F_k(\gamma) \\ K_1^{(k)} &= 1/2 [F_k(J+\gamma) - F_k(-J+\gamma)] \\ K_2^{(k)} &= 1/2 [F_k(J+\gamma) + F_k(-J+\gamma)] - F_k(\gamma) \end{aligned} \quad (72)$$

where $k = 1, 2$.

In order to construct the system of linear equations, all necessary coefficients can be obtained with the help of Eqs. (68), (71) and (72). We can investigate all phase diagrams by solving the system by following the procedure given in Section 3.

The phase diagrams in $(D/J - k_B T_c/J)$ plane for honeycomb ($z = 3$) lattice can be seen in Fig. 4.1 in comparison with DA and EFT. As seen in Fig. 4.1 the three approximation gives qualitatively same phase diagrams but we should call attention to quantitative differences. While EFT gives slightly lower critical temperatures than DA, the present formulation gives lower critical temperatures than EFT for all crystal field values. The quantitative differences can be seen in Table

4.1 for critical temperatures at $D = 0$ and Table 4.1 for tricritical points $(D_t/J, k_B T_t/J)$, at which the second order phase transition line and first order transition line meets. The D_t/J values are almost equal for three formulations, however $k_B T_t/J$ values of DA and EFT are close to each other, while the present formulation gives lower values for the $k_B T_t/J$. This fact can be seen in Table 4.1. The superiority of the present formulation can also be seen in the numerical values of critical temperatures in the limit $D \rightarrow \infty$, which are just the critical temperatures of the corresponding spin-1/2 system [76].

The phase diagrams in the $(D/J - k_B T_c/J)$ plane for the various three dimensional lattices ($z = 6$ simple cubic and $z = 8$ body centered lattices) can be seen in Fig. 2 in comparison with DA, as a limiting case of the bond diluted system, i.e. $c = 1$. Since the quantitative and qualitative relations of the phase diagrams of the EFT formulation and present formulation for these lattices are the same as $z = 3$ phase diagram, EFT diagrams have not been shown for this lattices.

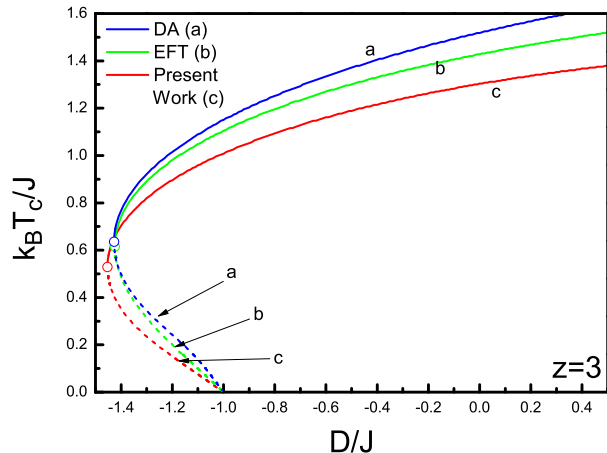


Figure 1: The phase diagram of the pure system for $z = 3$ in $(D/J - k_B T_c/J)$ plane. Solid and dotted curves correspond to the second and first order phase transitions, respectively and solid circles denote the tricritical points.

Table 1: The critical temperatures of the pure system with zero magnetic field obtained by DA, EFT and the present work for the crystal field $D/J = 0$.

Lattice	DA	EFT	Present Work
3	1.519	1.428	1.302
4	2.187	2.114	1.952
6	3.519	3.466	3.265
8	4.849	4.809	4.587
12	7.512	7.484	7.241

4.2 Bond Diluted System

Let us treat the bond dilution problem for homogenous crystal fields (i.e $P_D(D_i) = \delta(D_i - D)$ for all i) and zero magnetic field (i.e $H_i = 0$ for all lattice sites). For this system we assume that the nearest-neighbor interactions are randomly distributed on the lattice sites according to probability

Table 2: The tricritical points ($D_t/J, k_B T_t/J$) of the pure system with zero magnetic field obtained by DA, EFT and present work.

Lattice	DA		EFT		Present	
	D_t	$k_B T_t/J$	D_t	$k_B T_t/J$	D_t	$k_B T_t/J$
3	1.427	0.632	1.424	0.614	1.452	0.528
4	1.890	0.949	1.887	0.946	1.912	0.846
6	2.816	1.581	2.814	1.583	2.841	1.454
8	3.741	2.220	3.741	2.217	3.768	2.056
12	5.594	3.480	5.593	3.482	5.621	3.315

distribution function

$$P_J(J_{ij}) = (1 - c) \delta(J_{ij}) + c \delta(J_{ij} - J) \quad (73)$$

where c is the concentration of closed bonds and $0 < c \leq 1$.

Since the crystal field and the magnetic field distributions are same as the pure system with zero magnetic field, the functions which are used for calculating coefficients for this system will be (68). The configurational average in Eq. (58),

$$C_{np}^{(k)} = \binom{z}{n} \binom{n}{p} \prod_{\delta=1}^p \int dJ_{0\delta} P_J(J_{0\delta}) \sinh(J_{0\delta} \nabla) \prod_{\delta=p+1}^n \int dJ_{0\delta} P_J(J_{0\delta}) [\cosh(J_{0\delta} \nabla) - 1] F_k(x)|_{x=0} \quad (74)$$

can be taken by using Eq. (73) in Eq. (74) and it gives

$$C_{np}^{(k)} = \binom{z}{n} \binom{n}{p} [c \sinh(J \nabla)]^p [c \cosh(J \nabla) - c]^{n-p} F_k(x)|_{x=0} \quad (75)$$

where $k = 1, 2$.

Applying the same procedure between Eqs. (69)-(71) to Eq. (75) gives

$$C_{np}^{(k)} = \binom{z}{n} \binom{n}{p} c^n \sum_{r=0}^{n-p} \sum_{i=0}^r \sum_{j=0}^p \binom{n-p}{r} \binom{r}{i} \binom{p}{j} \frac{1}{2^{r+p}} (-1)^{n-p-r+j} F_k[(r+p-2i-2j)J]. \quad (76)$$

By the same way, the coefficients in Eq. (58) can be written for this system as

$$\begin{aligned} K_0^{(k)} &= F_k(\gamma) \\ K_1^{(k)} &= c/2 [F_k(J + \gamma) - F_k(-J + \gamma)] \\ K_2^{(k)} &= c/2 [F_k(J + \gamma) + F_k(-J + \gamma)] - c F_k(\gamma) \end{aligned} \quad (77)$$

where $k = 1, 2$.

As an example of the results obtained within the present formulation in comparison with the results of DA, let us investigate the phase diagram for simple cubic and body centered cubic lattices in ($D/J - k_B T_c/J$) plane for the bond diluted system. The general effect of dilution of the bonds in spin-1 BC model is to decrease the critical temperatures. At a certain bond concentration value, namely bond percolation threshold (c^*), ordered state is completely destroyed and the system cannot exhibit a ferromagnetic phase. Bond percolation threshold value depends on lattice type for a given Hamiltonian and for $c < c^*$ system shows no phase transition at all.

It can be seen in Fig. 2 that, T_c values of our formulation is lower than those obtained by DA for all values of c , as in the pure system with zero magnetic field. This means higher bond percolation threshold c^* value for the present formulation with respect to the DA. This can be seen in Table 4.2. Since the T_c values of the present formulation is closer to exact ones in comparison

with DA, it is expected that this is also true for the values of c^* . In Table 4.2 we see that, the difference in the bond percolation threshold values obtained in the present work and DA and EFT gets smaller as the coordination number increases.

For $D \rightarrow \infty$ it is obvious that the system behaves like spin-1/2 system. This implies that, the phase diagrams in Fig. 2 are getting closed on the right side, after the bond percolation threshold value of the corresponding spin-1/2 system. In general, since the calculation of the correlations within the EFT gives closer T_c or c^* value in comparison with the exact results than DA or results of other EFT approximations, we can say that c value at which phase diagrams getting closed on the right side in Fig. 2 (i.e corresponding S - 1/2 bond percolation threshold values) will be closer to exact ones also. This values can be found in our earlier work [76].

The order of the phase transition can be determined by examining the variation of the order parameter with temperature. As seen in Fig 3, we can conclude that all reentrant behaviors which emerge for the negative D/J values are of first order. In addition, magnetization behaviors for $D/J = 2.0$ for some selected c values can be seen in Fig. 3.

The other superiority of the present formulation is computability of the some thermodynamic functions which are related to the correlations (e.g. internal energy and thus specific heat). The internal energy and specific heat of simple cubic lattice for $D/J = 2.0$ with some selected bond concentration values are plotted in Fig. 4. The behaviors of the internal energy and specific heat are more reasonable than DA results, since DA neglects all multi site spin correlations. Besides it gives zero internal energy after the critical temperature, which is physically impossible. One important point about the specific heat curves is, appearance of a hump as c decreases. While c decreases, peak of the specific heat gets smoother and gives place to a growing hump after the bond percolation threshold value c^* .

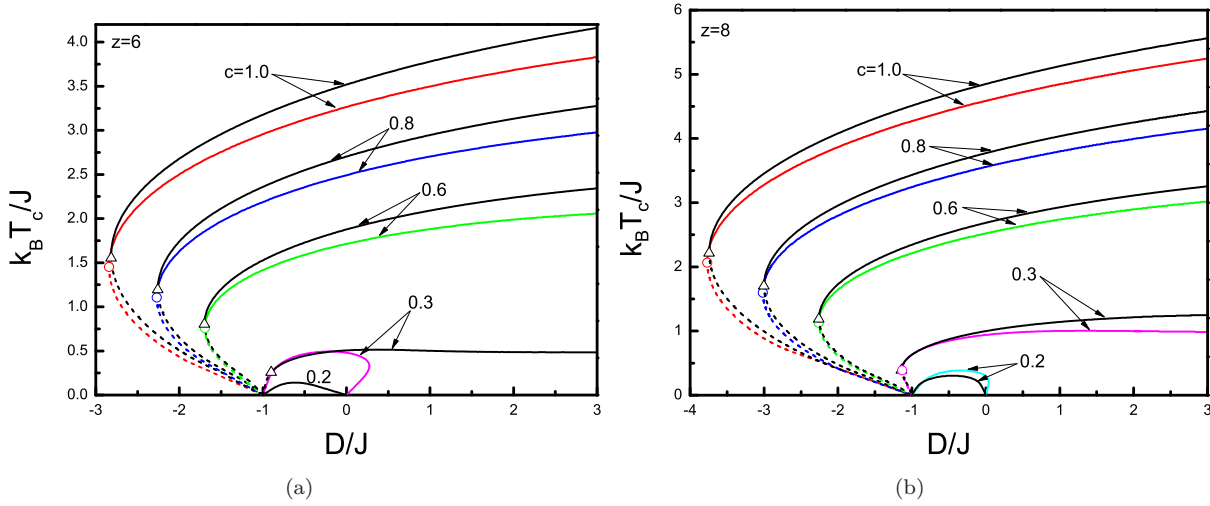


Figure 2: The phase diagrams of the bond diluted system in the $(D/J - k_B T_c/J)$ plane with selected values of the bond concentration c . Solid and dotted curves correspond to the second and first order phase transitions, respectively, while solid circles denote the tricritical points. For each c value, black curves are obtained by the DA and colored curves are obtained by the present formulation.

4.3 Crystal Field Dilution Problem

As a final investigation let us investigate the system with diluted crystal fields. The random crystal fields are distributed to lattice sites according to given probability distribution function,

$$P_D(D_i) = (1 - p) \delta(D_i) + p \delta(D_i - D) \quad (78)$$

The system is under zero magnetic field and there is no bond dilution, i.e $H_i = 0$ and $J_{ij} = J$ for all i, j . This changes only the functions defined in Eq. (10). Using $P_H(H_i) = \delta(H_i)$ and Eq. (78)

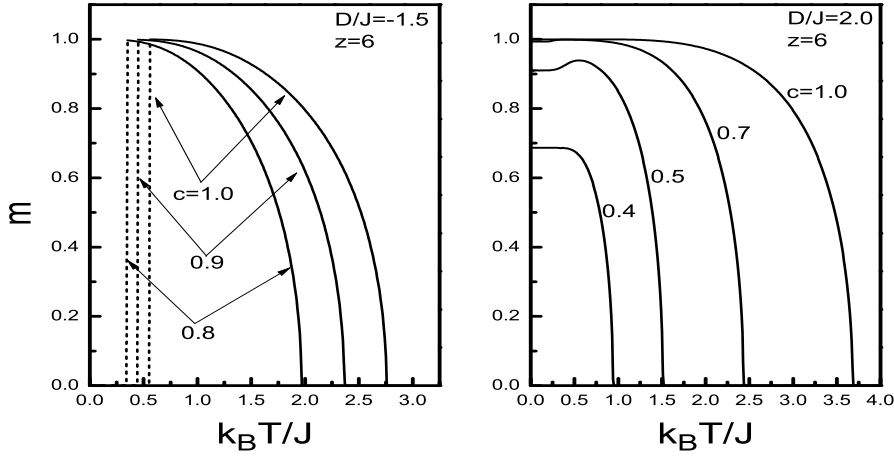


Figure 3: The variation of the magnetization with the temperature for the bond diluted simple cubic lattice, with selected values of the bond concentration c .

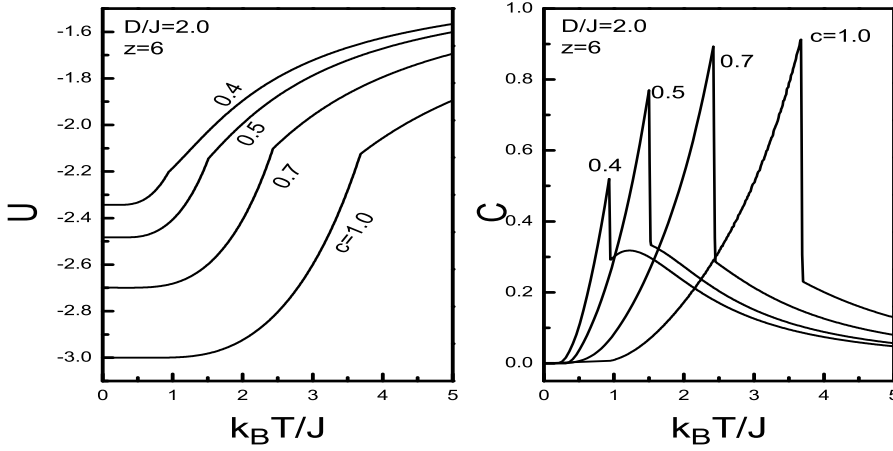


Figure 4: The variation of the internal energy (U) and the specific heat (C) with the temperature for the bond diluted simple cubic lattice, with selected values of the bond concentration c .

in Eq. (10) will give

$$F_1(x) = \frac{2(1-p) \sinh(\beta x)}{2 \cosh(\beta x) + 1} + \frac{2p \sinh(\beta x)}{2 \cosh(\beta x) + \exp(-\beta D)} \quad (79)$$

$$F_2(x) = \frac{2(1-p) \cosh(\beta x)}{2 \cosh(\beta x) + 1} + \frac{2p \cosh(\beta x)}{2 \cosh(\beta x) + \exp(-\beta D)} \quad (80)$$

By using Eqs. (76) and (77) with $c = 1$ and Eqs. (79) and (80) we can construct the system of linear equations.

The distribution function given in Eq. (78) distributes crystal fields in p percentage of lattice sites D and remaining $1 - p$ percentage of lattice sites zero. Because of this, $p = 0$ phase diagrams in $(D/J - k_B T_c/J)$ plane will be the line parallel to the D/J axis and the value of the $k_B T_c/J$ is just the critical temperature of the pure system (i.e. system with no external magnetic field, homogenous distributed D and J) at $D = 0$. The phase diagrams in this plane will evolve towards the phase diagram of pure system in the same plane, when concentration value goes to $p = 1$. During this evolution, up to a certain $p = p^*$ value, phase diagrams will not intersect D/J axis on the left side of the $(D/J - k_B T_c/J)$ plane (i.e. region with negative D/J values). This means that, system is in the ordered phase at low temperatures for all D/J values for $p < p^*$.

Table 3: The bond percolation threshold values (c^*) of the bond diluted system with zero magnetic field, obtained by DA, EFT and present formulation for different lattices.

Lattice	DA	EFT	Present Work
3	0.333	0.347	0.391
4	0.249	0.256	0.282
6	0.166	0.168	0.181
8	0.124	0.126	0.133
12	0.083	0.084	0.086

The phase diagrams of the lattices with coordination numbers $z = 6, 8$ (simple cubic and body centered lattice respectively) can be seen in Fig. 5 in comparison with DA phase diagrams. Again we can see from Fig. 5 that the T_c values of introduced formulation are lower than those obtained by DA. This makes the critical concentration value p^* lower than that of DA. The p^* values for different lattices can be seen in Table 4.3.

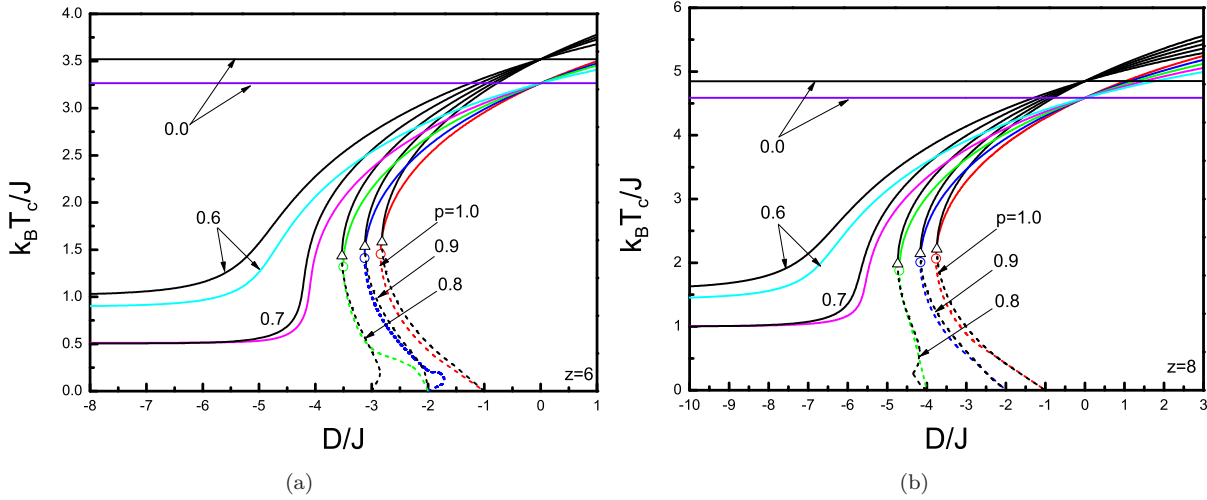


Figure 5: The phase diagrams of the crystal field diluted system in $(D/J - k_B T_c/J)$ plane with selected values of the crystal field dilution parameter (p). Solid and dotted curves correspond to the second and first order phase transitions, respectively and solid circles denote the tricritical points. For each p value, black curves are obtained by the DA and colored curves are obtained by the present formulation.

5 Conclusion

In this work we have presented a general formulation for the spin-1 Ising system with BC model with nearest neighbor interactions. Within the introduced formulation we can calculate the multi site spin correlations in a representative manner. This capability shows its superiority in the results of the critical temperatures and other critical values (e.g. bond percolation threshold values) and behavior of thermodynamic functions (e.g specific heat) as a function of the temperature. Our formulation can be applied to the systems with quenched disordered systems. These type of

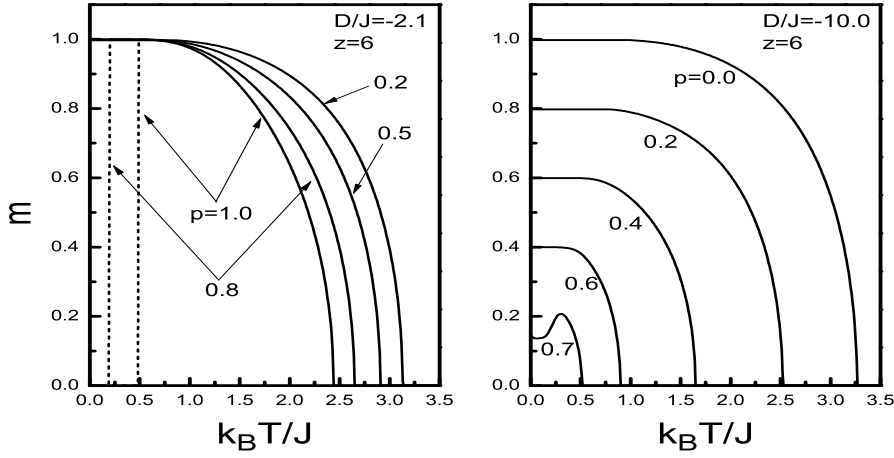


Figure 6: The variation of the magnetization with temperature for the system on a simple cubic lattice, with selected values of the crystal field dilution parameter p .

Table 4: The critical crystal field concentration values of crystal field diluted system with zero magnetic field, obtained by DA,EFT and present formulation for different lattices.

Lattice	DA	EFT	Present Work
3	0.484	0.476	0.379
4	0.604	0.611	0.544
6	0.729	0.745	0.718
8	0.794	0.810	0.802
12	0.861	0.875	0.892

disorders in real materials can be extremely important since they can induce important macroscopic effects on the thermal and magnetic properties of the material. The systems corresponding to these systems cannot be solved exactly in most cases. Hence, some approximations must be done. But, including approximations in the calculations means that loosing accuracy of the results. This is a disadvantageous situation from the experimental point of view of the material science, although making approximations simplify the solutions. Thus, it is desired to improve approximations as much as possible to give more closer solutions to exact ones.

As shown in Section 4, our formulation gives the best approximated solutions of BC model for the spin-1 system within the framework of EFT. Besides, it can give more reasonable behaviors of some thermodynamic functions (e.g. internal energy, specific heat) in comparison to the other EFT formulations.

On the other hand we believe that, it is very important to develop a general formulation which is capable of applying to the related systems with arbitrary coordination numbers. In conclusion, we hope that the formulation and results obtained in this work may be beneficial form both theoretical and experimental point of view.

6 Appendix A

During the derivation process of correlations in a sequential order given in Eq. (21) we envision that each coefficient A_{np} in the operators moves to another correlation as a multiplier of it at each

step of the derivation. Thus, derivation process generate a movement of the coefficients A_{np} . We can track the 'path' all coefficients during this derivation process and determine the place of all coefficients at the k_{th} step of this movement.

According to Eq. (28) there are three possible 'ways' for a coefficient. Let us call movements in these three possible different ways as movements in regime 1, regime 2 and regime 3. In the regime 1, movement goes on according to the first line of the right hand side of Eq. (28). Similarly in the regime 2 movement goes as the the second line of the right hand side of Eq. (28) and the movement in regime 3 goes as third line.

In Eq. (29), we give the path of the coefficient A_{00} from the beginning to the end of the movement. This movement always stays in the regime 1 since at each step, the step number k satisfies $k > l + m$, where indices l, m defines the term as $s_\delta^{(l)} q_\delta^{(m)}$ which is the starting point of the k_{th} step.

Now we give the paths of the remaining coefficients according to this reasoning. At the beginning of the movement, the coefficients A_{n0} ($n \neq 0$) are placed at the $s_\delta^{(0)} q_\delta^{(n)}$. For the first step ($k = 1$) since $0 < k \leq n$ the movement starts in the regime 2. From Eq. (28) we can generate the path of these coefficients while the derivation process in a sequential order given in Eq. (21). The first n step of these coefficients in the regime 2 are given as follows

$$s_\delta^{(0)} q_\delta^{(n)} \rightarrow s_\delta^{(1)} q_\delta^{(n-1)} \rightarrow \dots \rightarrow s_\delta^{(n)} q_\delta^{(0)} \quad (81)$$

We see from Eq. (81) that at the end of the first n step, the coefficient A_{n0} arrives at $s_\delta^{(n)} q_\delta^{(0)}$. After then, the movement goes on in the regime 1 since the step $k = n$ starts to fulfill the condition $k > n$

$$s_\delta^{(n)} q_\delta^{(0)} \rightarrow s_\delta^{(n+1)} q_\delta^{(0)} \rightarrow \dots \rightarrow s_\delta^{(z)} q_\delta^{(0)} \quad (82)$$

Thus for the coefficient A_{n0} , we can conclude that for the steps $k \leq n$ it is at the place $s_\delta^{(k)} q_\delta^{(n-k)}$ (Eq. (81)), and for the steps $k > n$ it is at the place $s_\delta^{(k)} q_\delta^{(0)}$ (Eq. (82)) i.e we arrive Eq. (31).

Remaining coefficients A_{np} ($n, p > 0$) are placed at $s_\delta^{(p)} q_\delta^{(n-p)}$ at the beginning. Since for the first step $k = 1 \leq n$ is satisfied, then the coefficient A_{np} starts its movement in the regime 3 and it stays in this regime until the step r which is $p - r = r$ i.e. $r = p/2$ according to Eq. (28).

$$s_\delta^{(p)} q_\delta^{(n-p)} \rightarrow s_\delta^{(p-1)} q_\delta^{(n-p+1)} \dots \rightarrow s_\delta^{(p-r)} q_\delta^{(n-p+r)} \quad (83)$$

If the index p , which is the number of s_δ in the term $s_\delta^{(p)} q_\delta^{(n-p)}$ is odd, then we have $r = (p+1)/2$. Thus we can say that, the coefficient A_{np} ($n, p > 0$) is at $s_\delta^{(p-k)} q_\delta^{(n-p+k)}$, for the end of the step $k \leq p/2$ (for even p) and for the end of the step $k \leq (p+1)/2$ (for odd p) i.e. the first lines of the right hand sides of Eqs. (32) and (33).

At the end of the movement given in Eq. (83) the coefficient A_{np} ($n, p > 0$) is at the place $s_\delta^{(p-r)} q_\delta^{(n-p+r)}$. After than the movement goes in the second regime since the step number k becomes greater than $p - r$. The movement stays in the regime 2 for t steps as,

$$s_\delta^{(p-r)} q_\delta^{(n-p+r)} \rightarrow s_\delta^{(p-r+1)} q_\delta^{(n-p+r-1)} \dots \rightarrow s_\delta^{(p-r+t)} q_\delta^{(n-p+r-t)} \quad (84)$$

Since in the regime 3 and regime 2 keeps the places of the coefficients $s_\delta^{(l)} q_\delta^{(m)}$ to be $l + m$ constant, at a certain step, step number will catch $l + m$ then the movement goes on a regime 1 according to the first line of the right hand side of Eq. (28). This step number satisfies $r + t = n$. Then while for even p this condition reads $p/2 + t = n$, for odd p reads $(p+1)/2 + t = n$. We find that for even p , $t = n - p/2$ and for odd p , $t = n - (p+1)/2$. Thus the movement given in Eq. (84) ends with $s_\delta^{(n)} q_\delta^{(0)}$ for even p and $s_\delta^{(n-1)} q_\delta^{(1)}$ for odd p . We conclude from Eq. (84) that for even p the coefficient A_{np} ($n, p > 0$) is at the place $s_\delta^{(k)} q_\delta^{(n-k)}$ and for the odd p at the place $s_\delta^{(k-1)} q_\delta^{(n-k+1)}$ for the end of the step $r < k \leq r + t$. These results are the second lines of the right hand sides of Eqs. (32) and (33).

Now the movement is in the regime 1 and according to Eq. (28) it goes as

$$s_\delta^{(n)} q_\delta^{(0)} \rightarrow s_\delta^{(n+1)} q_\delta^{(0)} \dots \rightarrow s_\delta^{(z)} q_\delta^{(0)} \quad (85)$$

$$s_{\delta}^{(n-1)} q_{\delta}^{(1)} \rightarrow s_{\delta}^{(n)} q_{\delta}^{(1)} \dots \rightarrow s_{\delta}^{(z)} q_{\delta}^{(1)} \quad (86)$$

From Eqs. (85) and (86) we can say that the coefficient A_{np} ($n, p > 0$) is at the place $s_{\delta}^{(k)} q_{\delta}^{(0)}$ for even p and at the place $s_{\delta}^{(k-1)} q_{\delta}^{(1)}$ for odd p for the end of the step $k > n$. This corresponds to the third lines of the right hand sides of the Eqs. (32) and (33) and this completes our derivation of Eqs. (31)-(33).

7 Appendix B

As explained in Section 3, Eqs. (42)-(53) have to give correct spin-1/2 equivalents in the limits $D \rightarrow \infty$ and $q_i \rightarrow 1$ where $i = 0, 1, \dots, z$.

The central spin and the perimeter spin averages for the nearest neighbor spin-1/2 Ising model with random bond and random magnetic field distribution can be given as [76]

$$\langle \langle s_0 \rangle \rangle_r = \left\langle \left\langle \Theta_0^{(1/2)} \right\rangle \right\rangle_r f(x)|_{x=0} \quad (87)$$

$$\langle \langle s_1 \rangle \rangle_r = \left\langle \left\langle \Phi_0^{(1/2)} \right\rangle \right\rangle_r f(x + \gamma)|_{x=0} \quad (88)$$

where

$$g(x, H_i) = \tanh [\beta (x + H_i)], \quad f(x) = \int dH_i P_H (H_i) g(x, H_i) \quad (89)$$

The operators in Eqs. (87) and (88) can be written as

$$\Theta_0^{(1/2)} = \sum_{n=0}^z A_n^{(1/2)} s_{\delta}^{(n)} \quad (90)$$

$$\Phi_0^{(1/2)} = B_0^{(1/2)} + B_1^{(1/2)} s_0 \quad (91)$$

where the coefficients are given by

$$\begin{aligned} A_n^{(1/2)} &= \binom{z}{n} \prod_{\delta=1}^n \sinh (J_{0\delta} \nabla) \prod_{\delta=n+1}^z \cosh (J_{0\delta} \nabla), \quad n = 0, 1, \dots, z \\ B_0^{(1/2)} &= \cosh (J_{0\delta} \nabla) \\ B_1^{(1/2)} &= \sinh (J_{0\delta} \nabla) \end{aligned} \quad (92)$$

With the operators

$$\Phi_k^{(1/2)} = B_0^{(1/2)} s_{\delta}^{(k)} + B_1^{(1/2)} s_0 s_{\delta}^{(k)} \quad (93)$$

$$s_0 \Phi_k^{(1/2)} = B_0^{(1/2)} s_0 s_{\delta}^{(k)} + B_1^{(1/2)} s_{\delta}^{(k)} \quad (94)$$

$$\Theta_k^{(1/2)} = s_{\delta}^{(k-1)} \sum_{n=0}^{k-1} A_{2n+1}^{(1/2)} + s_{\delta}^{(k)} \sum_{n=0}^k A_{2n}^{(1/2)} + \sum_{n=2k+1}^z A_n^{(1/2)} s_{\delta}^{(n-k)} \quad (95)$$

we can obtain spin-1/2 multi spin correlations as

$$\left\langle \left\langle s_{\delta}^{(k)} \right\rangle \right\rangle_r = \left\langle \left\langle \Phi_{k-1}^{(1/2)} \right\rangle \right\rangle_r f(x + \gamma)|_{x=0} \quad (96)$$

$$\left\langle \left\langle s_0 s_{\delta}^{(k)} \right\rangle \right\rangle_r = \left\langle \left\langle \Theta_k^{(1/2)} \right\rangle \right\rangle_r f(x)|_{x=0} \quad (97)$$

$$\left\langle \left\langle s_0 s_{\delta}^{(k)} \right\rangle \right\rangle_r = \left\langle \left\langle s_0 \Phi_{k-1}^{(1/2)} \right\rangle \right\rangle_r f(x + \gamma)|_{x=0} \quad (98)$$

Now, we can see from Eq. (9)

$$\lim_{D \rightarrow \infty} G_1(x, H_i) = g(x, H_i), \quad \lim_{D \rightarrow \infty} G_2(x, H_i) = 1 \quad (99)$$

and then

$$\lim_{D \rightarrow \infty} F_1(x) = f(x), \quad \lim_{D \rightarrow \infty} F_2(x) = 1 \quad (100)$$

where $f(x), g(x)$ are the spin-1/2 functions which are defined in Eq. (89).

Let us label all operators and coefficients related to the spin-1 system with superscript (1). With a little combinatorics one can show that

$$\sum_{n=p}^z A_{np}^{(1)} = A_p^{(1/2)} \quad (101)$$

is valid for the relation between the coefficients of spin-1/2 and spin-1 Ising system which is given in Eq. (27).

Let us look at the $q_i \rightarrow 1$, ($i = 0, 1, \dots, z$) limits of all necessary operators for deriving multi site correlations and fundamental equalities of spin-1 Ising system which are given in Eqs. (26), (34), (36), (40) and (41). We start with Eq. (26).

$$\lim_{q_i \rightarrow 1} \Theta_{00}^{(1)} = \sum_{n=0}^z \sum_{p=0}^n A_{np}^{(1)} s_\delta^{(p)} = \sum_{p=0}^z s_\delta^{(p)} \sum_{n=p}^z A_{np}^{(1)} \quad (102)$$

According to Eq. (101) this will give Eq. (90). Thus

$$\lim_{q_i \rightarrow 1} \Theta_{00}^{(1)} = \Theta_0^{(1/2)} \quad (103)$$

The spin-1/2 limit of the operator given in Eq. (34) by using Eq. (35) can be obtained as

$$\begin{aligned} \lim_{q_i \rightarrow 1} \Theta_{k,0}^{(1)} &= \left(\sum_{n=0}^z A_{n0}^{(1)} + \sum_{n=1}^{k-1} \sum_{p=2}^{n''} A_{np}^{(1)} + \sum_{n=k}^z \sum_{p=2}^{2k-1''} A_{np}^{(1)} \right) s_\delta^{(k)} \\ &+ \left(\sum_{n=1}^{k-1} \sum_{p=1}^{n'} A_{np}^{(1)} + \sum_{n=k}^z \sum_{p=1}^{2k-2'} A_{np}^{(1)} \right) s_\delta^{(k-1)} + \sum_{n=k, p=2k-1}^z \sum_{n=p}^n A_{np}^{(1)} s_\delta^{(p-k)} \end{aligned} \quad (104)$$

by rearranging the sums we get

$$\lim_{q_i \rightarrow 1} \Theta_{k,0}^{(1)} = s_\delta^{(k)} \sum_{p=0}^{2k''} \sum_{n=p}^z A_{np}^{(1)} + s_\delta^{(k-1)} \sum_{p=1}^{2k-1'} \sum_{n=p}^z A_{np}^{(1)} + \sum_{p=2k+1}^z \sum_{n=p}^z A_{np}^{(1)} s_\delta^{(p-k)} \quad (105)$$

According to Eq. (101), the right hand side of Eq. (105) is equal to the right hand side of Eq. (95), then

$$\lim_{q_i \rightarrow 1} \Theta_{k,0}^{(1)} = \Theta_k^{(1/2)} \quad (106)$$

The spin-1/2 limit of the operator given in Eq. (36) is

$$\begin{aligned} \lim_{q_i \rightarrow 1} \Theta_{k,m}^{(1)} &= \left(\sum_{n=0}^z A_{n0}^{(1)} + \sum_{n=1}^{k-1} \sum_{p=2}^{n''} A_{np}^{(1)} + \sum_{n=k}^z \sum_{p=2}^{2k-1''} A_{np}^{(1)} \right) s_\delta^{(k-m)} \\ &+ \left(\sum_{n=1}^{k-1} \sum_{p=1}^{n'} A_{np}^{(1)} + \sum_{n=k}^z A_{n,2k-1}^{(1)} + \sum_{n=k}^z \sum_{p=1}^{2k-2'} A_{np}^{(1)} \right) s_\delta^{(k-m+1)} + \sum_{n=k, p=2k}^z \sum_{n=p}^n A_{np}^{(1)} s_\delta^{(p-k-m)} \end{aligned} \quad (107)$$

We can see from Eq. (95) that, Θ_{k-m} contains $s_\delta^{(k-m-1)}$, instead of $s_\delta^{(k-m+1)}$ in Eq. (107), then we can conclude that

$$\lim_{q_i \rightarrow 1} \Theta_{k,m}^{(1)} \neq \Theta_{k-m}^{(1/2)} \quad (108)$$

The spin-1/2 limit of Eq. (40) is given by

$$\lim_{q_i \rightarrow 1} \Phi_{k,m}^{(1)} = \left(B_0^{(1)} + B_2^{(1)} \right) s_\delta^{(k-m)} + B_1^{(1)} s_0 s_\delta^{(k-m)} \quad (109)$$

With the help of Eq. (38), we can conclude from Eq. (109) with comparing (93) that

$$\lim_{q_i \rightarrow 1} \Phi_{k,m}^{(1)} = \Phi_{k-m}^{(1/2)} \quad (110)$$

Similarly we can obtain the results for the operators given in Eq. (41) as

$$\begin{aligned} \lim_{q_i \rightarrow 1} s_0 \Phi_{k,m}^{(1)} &= s_0 \Phi_{k-m}^{(1/2)} \\ \lim_{q_i \rightarrow 1} q_0 \Phi_{k,m}^{(1)} &= \Phi_{k-m}^{(1/2)} \end{aligned} \quad (111)$$

Now we can look at the spin-1/2 limits of expressions given in Eqs. (42)-(53), in order to decide whether they have correct limits or not. Let us start with Eq. (42). Whereas the limit of the left hand side of Eq. (42) is

$$\lim_{q_i \rightarrow 1} \left\langle \left\langle s_\delta^{(k-m)} q_\delta^{(m)} \right\rangle \right\rangle_r = \left\langle \left\langle s_\delta^{(k-m)} \right\rangle \right\rangle_r \quad (112)$$

the limit of the right hand side of Eq. (42) can be obtained as by using Eqs. (100) and (110)

$$\lim_{q_i \rightarrow 1} \left\langle \left\langle \Phi_{k-1,m}^{(1)} \right\rangle \right\rangle_r F_1(x+\gamma)|_{x=0} = \left\langle \left\langle \Phi_{k-1-m}^{(1/2)} \right\rangle \right\rangle_r f(x+\gamma)|_{x=0} \quad (113)$$

Then with the help of Eqs. (112) and (113), we can obtain for (42)

$$\left\langle \left\langle s_\delta^{(k-m)} \right\rangle \right\rangle_r = \left\langle \left\langle \Phi_{k-1-m}^{(1/2)} \right\rangle \right\rangle_r f(x+\gamma)|_{x=0} \quad (114)$$

This is nothing but Eq. (96) for the correlation $\left\langle \left\langle s_\delta^{(k-m)} \right\rangle \right\rangle_r$ for spin-1/2 system. Thus we can conclude that expression given in Eq. (42) has correct spin-1/2 limit.

In a similar manner, Eq. (43) has spin-1/2 limit as

$$\left\langle \left\langle s_\delta^{(k-m)} \right\rangle \right\rangle_r = \left\langle \left\langle \Phi_{k-m}^{(1/2)} \right\rangle \right\rangle_r 1 \quad (115)$$

From Eq. (93) we can write Eq. (115) as

$$\left\langle \left\langle s_\delta^{(k-m)} \right\rangle \right\rangle_r = \left\langle \left\langle s_\delta^{(k-m)} \right\rangle \right\rangle_r B_0^{(1/2)} 1 + \left\langle \left\langle s_0 s_\delta^{(k-m)} \right\rangle \right\rangle_r B_1^{(1/2)} 1 \quad (116)$$

and from Eq. (92), we have

$$\left\langle \left\langle s_\delta^{(k-m)} \right\rangle \right\rangle_r = \left\langle \left\langle s_\delta^{(k-m)} \right\rangle \right\rangle_r \quad (117)$$

i.e tautology.

If we continue to work on the spin-1/2 limits of the expressions given in Eqs. (44)-(53), we can see that the expressions (44), (45), (48), (49) and (53) gives the correct limits as in (114) for the expression (42) and the expressions (47), (50), (51) and (52) gives tautology as in (117) for the expression (43). The only expression gives false spin-1/2 limit is (46).

References

- [1] M. Blume, Phys.Rev. 141 (1966) 517.
- [2] H. W. Capel, Physica 32 (1966) 966.
- [3] I. D. Lawrie, S. Sarback (1988) Phase Transitions and Critical Phenomena vol 9 ed C. Domb, J. L. Lebowitz (London: AcademicPress).
- [4] A. Maritan, M. Cieplak, M. R. Swift, F. Toigo, J. R. Banavar, Phys.Rev.Lett. 69 (1992) 221.
- [5] S. L. Lock, B. S. Lee, Phys. Status Solidi b 124 (1984) 593.
- [6] Y. Tanaka, N. Uryu, J. Phys. Soc. Japan 50 (1981) 1140.
- [7] J. G. Brankov , J. Przystawa, E. Praveczi, J. Phys. C: Solid State Phys. 5 (1972) 3387.
- [8] D. M. Saul, M. Wortis, D. Stauffer, Phys. Rev. B 9 (1974) 4964.
- [9] W. M. Ng, J. Barry, Phys. Rev. B 17 (1978) 3675.
- [10] J. W. Tucker, T. Balcerzak, M. Gzik, A. Sukiennicki, J. Magn. Magn. Mater. 187 (1998) 381.
- [11] A. K. Jain, D. P. Landau, Phys. Rev. B 22 (1980) 445.
- [12] J. D. Kimel, P. A. Rikvold, Y. L. Wang, Phys. Rev. B 45 (1992) 7237.
- [13] W. Selke, J. Yeomans, J. Phys. A: Math. Gen. 16 (1983) 2789.
- [14] D. P. Landau, R. H. Swendsen, Phys. Rev. B 33 (1985) 7700.
- [15] A. N. Berker, M. Wortis, Phys. Rev. B 14 (1976) 4946.
- [16] T. W. Burkhardt, Phys. Rev. B 14 (1976) 1196.
- [17] T. W. Burkhardt, H. J. F. Knops, Phys. Rev. B 15 (1977) 1602.
- [18] A. E. Siqueira, I. P. Fittipaldi, Physica A 138 (1986) 592.
- [19] K. G. Chakraborty, Phys. Rev. B 29 (1984) 1454.
- [20] X. F. Jiang, J. L. Li, J. L. Zhong, Phys. Rev. B 47 (1993) 827.
- [21] T. Kaneyoshi, I. Tamura, Phys. Rev. B 25 (1982) 4679.
- [22] T. Kaneyoshi, Acta Phys. Pol. A 83 (1993) 703.
- [23] T. Kaneyoshi, Physica A 269 (1999) 344.
- [24] T. Kaneyoshi, Physica A 269 (1999) 357.
- [25] T. Kaneyoshi, Physica A 286 (2000) 518.
- [26] A. Du ,Y. Q. Yu, H. J. Liu, Physica A 320 (2003) 387.
- [27] A. Du ,H. J. Liu, Y. Q. Yü, Phys. Status Solidi b 241 (2004) 175.
- [28] Q. Zhang, G. Wei, Z. Xin, Y. J. Liang, J. Magn. Magn. Mater. 280 (2004) 14.
- [29] T. Kaneyoshi, J. Phys. C 19 (1986) L557.
- [30] T. Kaneyoshi, J. Phys. C 21(1988) L679.
- [31] T. Kaneyoshi, J. Mielnicki, J. Phys. Condens. Matter 2 (1990) 8773.
- [32] T. Kaneyoshi, Phys. Status Solidi b 170 (1992) 313.

- [33] S. L. Yan, L. L. Deng, *Physica A* 308 (2002) 301.
- [34] A. Benyoussef, T. Biaz, M. Saber, M. Touzani, *J. Phys. C* 20 (1987) 5349.
- [35] M. E. S. Borelli, C. E. I. Carneiro, *Physica A* 230 (1996) 249.
- [36] C. E. I. Carneiro, V. B. Henriques, S. R. Salinas, *J. Phys. Condens. Matter* 1 (1989) 3687.
- [37] N. Boccara, A. El Kenz, M. Saber, *J. Phys. Condens. Matter* 1 (1989) 5721.
- [38] C. E. I. Carneiro, V. B. Henriques, S. R. Salinas, *J. Phys. A Math. Gen.* 23 (1990) 3383.
- [39] L. Bahmad, A. Benyoussef, A. El Kenz, *J. Magn. Magn. Mater.* 320 (2008) 397.
- [40] O. D. Salmon, J. R. Tapia, *J. Phys. A: Math. Theor.* 43 (2010) 125003.
- [41] C. Buzano, A. Maritan, A. Pelizzola, *J. Phys. Condens. Matter* 6 (1994) 327.
- [42] E. Albayrak, *Physica A* 390 (2011) 1529.
- [43] A. Benyoussef, H. Ez-Zahraouy, *J. Phys. Condens. Matter* 6 (1994) 3411.
- [44] V. Ilkovic, *Phys. Status Solidi b* 192 (1995) K7
- [45] I. Puha, H. T. Diep, *J. Magn. Magn. Mater.* 224 (2001) 85.
- [46] D. P. Lara, J. A. Plascak, *Physica A* 260 (1998) 443
- [47] N. S. Branco, B. M. Boechat, *Phys. Rev. B* 56 (1997) 11673
- [48] A. El-Atri, M. Saber, *Phys. Stat. Sol. b* 184 (1994) 187.
- [49] T. Bouziane, M. Saber, *J. Magn. Magn. Mater.* 321 (2009) 17.
- [50] L. L. Deng, S. L. Yan, *J. Magn. Magn. Mater.* 251 (2002) 138.
- [51] H. P. Dong, S. L. Yan, *Solid State Commun.* 139 (2006) 406.
- [52] X. F. Jiang, *J. Magn. Magn. Mater.* 134 (1994) 167.
- [53] T. Kaneyoshi, J. Mielnicki, T. Balcerzak, G. Wiatrowski, *Phys. Rev. B* 42 (1990) 4388.
- [54] T. Kaneyoshi, *J. Phys. Soc. Japan* 56 (1987) 933.
- [55] M. Kerouad, A. El-Atri, A. Ainane, M. Saber, *Phys. Stat. Sol. B* 195 (1996) 519.
- [56] E. F. Sarmiento, R. B. Muniz, S. B. Cavalcanti, *Phys. Rev. B* 36 (1987) 529.
- [57] J. W. Tucker, *J. Magn. Magn. Mater.* 132 (1994) 231.
- [58] X. F. Jiang, J. L. Zhong, J. L. Li, *J. Phys. Cond. Mat.* 6 (1994) 523.
- [59] C. Z. Yang, J. L. Zhong, *Phys. Stat. Sol. b* 153 (1989) 323.
- [60] C. Z. Yang, J. L. Zhong, X. F. Jiang, *Phys. Stat. Sol. B* 165 (1991) 539.
- [61] N. Benayad, A. Benyoussef, N. Boccara, *J. Phys. C Solid State Phys.* 18 (1985) 1899.
- [62] N. Benayad, A. Benyoussef, N. Boccara, *J. Phys. C Solid State Phys.* 21 (1988) 5417.
- [63] M. Kerouad, M. Saber, *Phys. Stat. Sol. b* 168 (1991) 597.
- [64] L. Tóth, M. Tóthová, *J. Phys. Condens. Matter.* 5 (1993) 121.
- [65] L. Tóth, M. Tóthová, *Phys. Stat. Sol. b* 167 (1991) K63.

- [66] J. B. Santos Filho, N. O. Moreno, D. F. de Albuquerque, A. S. de Arruda, *Physica B* 398 (2007) 294.
- [67] A. Roder, J. Adler, W. Janke, *Phys. Rev. Lett.* 80 (1998) 4697.
- [68] A. Bobák, M. Jascur, *J. Phys. Condens. Matter* 2 (1990) 10053.
- [69] L. Tóth, M. Tóthová, *IEEE Trans. Magn.* 30 (1994) 887.
- [70] L. Tóth, M. Tóthová, *Phys. Stat. Sol. b* 186 (1994) K19.
- [71] H. Dickinson, J. Yeomans, *J. Phys. C Solid State Phys.* 16 (1983) L34S.
- [72] F. C. SáBarreto, I. P. Fittipaldi, B. Zeks, *Ferroelectrics* 39 (1981) 1103.
- [73] Y. Yüksel, Ü. Akıncı H. Polat, *Phys. Scr.* 79 (2009) 045009
- [74] Ü. Akıncı, Y. Yüksel, H. Polat, *Physica A* 390 (2011) 541
- [75] Y. Yüksel, Ü. Akıncı H. Polat, doi:10.1016/j.physa.2011.12.060
- [76] Ü. Akıncı, arXiv:1112.4052